**Practice Questions for Midterm 1 – Math 1060Q – Fall 2014**

The following is a selection of problems to help prepare you for the first midterm exam. Please note the following:

- there may be mistakes – email steven.pon@uconn.edu if you find one.
- learning math is about more than just memorizing the steps to certain problems. You have to understand the concepts behind the math. In order to test your understanding of the math, you may see questions on the exam that are unfamiliar, but that rely on the concepts you've learned. Therefore, you should concentrate on learning the underlying theory, not on memorizing steps without knowing *why* you're doing each step.

1. Review your algebra skills! Although there will not be explicit "algebra" questions on the exam, your grade on the exam will be hurt significantly if you make algebraic mistakes. You can go back to the Algebra Exam Review for some practice problems.

2. Let $f(x) = 5x - 2$ and $g(x) = x^2 + 3x$.

   (a) What is $(f \circ g)(3)$?

   (b) What is $g(\sqrt{x})$?

   (c) What is $g(1 + h)$?

   (d) What is $f^{-1}(2)$?

   (e) Is $g$ one-to-one? Why or why not?

   (f) Which of the following points is on the graph of $f^{-1}$?
   (a) (0,3) (b) (2,1) (c) (0,0) (d) (3,1)

   (g) Compute a formula for $(g \circ f)(x)$ and simplify it.

   (h) What is the range of $g$?

   (i) Compute a formula for $f^{-1}(x)$.

   (j) What is the domain of $f/g$?

   (k) What is the domain of $g/f$?

   (l) What is the domain of $f \circ g$?

   (m) What is the domain of $g \circ f$?

**Solution:**

(a) $(f \circ g)(3) = f(g(3)) = f(18) = 88$

(b) $g(\sqrt{x}) = \sqrt{x}^2 + 3\sqrt{x} = x + 3\sqrt{x}$

(c) $g(1 + h) = (1 + h)^2 + 3(1 + h) = 1 + 2h + h^2 + 3 + 3h = h^2 + 5h + 4$

(d) $f^{-1}(2) = y \iff f(y) = 2$, and so $y = \frac{4}{5}$

(e) No, $g$ is not 1-1. For example, $g(0) = 0$ and $g(-3) = 0$.

(f) If $(a, b)$ is on the graph of $f^{-1}$, that means $f^{-1}(a) = b$, which means $f(b) = a$. So we can test using $f$. $f(3) = 13$, so $(13, 3)$ should be on the graph of $f^{-1}$, not $(0, 3)$. $f(1) = 3$. $f(0) = -2$. $f(1) = 3$...hey, we found one. If $f(1) = 3$, then $f^{-1}(3) = 1$, so $(3, 1)$ is on the graph of $f^{-1}$.

(g) $(g \circ f)(x) = (5x - 2)^2 + 3(5x - 2) = 25x^2 - 20x + 4 + 15x - 6 = 25x^2 - 5x - 2$

(h) You can complete the square to write $g$ in a more useful form: $g(x) = (x + \frac{3}{2})^2 - \frac{9}{4}$. Since $g$ is an upward-opening parabola with a vertex at $(-\frac{3}{2}, -\frac{9}{4})$, its range is $[-\frac{9}{4}, \infty)$.

(i) $x = 5y - 2 \implies x + 2 = 5y \implies y = \frac{x+2}{5}$, so $f^{-1}(x) = \frac{x+2}{5}$

(j) Domain of $f$ is $(-\infty, \infty)$ and domain of $g$ is $(-\infty, \infty)$, so all we need to worry about is when $g(x) = 0$, which is when $x^2 + 3x = 0$, which is when $x(x+3) = 0$, which is at 0 and $-3$. Therefore, domain of $f/g$ is $(-\infty, -3) \cup (-3, 0) \cup (0, \infty)$.

(k) Domain of $g/f$ is all real numbers except for when $5x - 2 = 0$, i.e., domain is $(-\infty, \frac{2}{5}) \cup (\frac{2}{5}, \infty)$.

(l) $(-\infty, \infty)$

(m) $(-\infty, \infty)$

3. Let $f(x) = \sqrt{2-x}$ and $g(x) = 4/x$. Find a formula for each of $f+g$, $f \cdot g$, $f/g$, $g/f$, $f \circ g$, and $g \circ f$. In the first four cases, find the domain of the function.

**Solution:**
$(f+g)(x) = \sqrt{2-x} + \frac{4}{x}$, domain $= \{x \mid x \neq 0, x \leq 2\}$, or in other words, $(-\infty, 0) \cup (0, 2]$. $(f \cdot g)(x) = \frac{4\sqrt{2-x}}{x}$, domain $= (-\infty, 0) \cup (0, 2]$. $(f/g)(x) = \frac{x\sqrt{2-x}}{4}$, domain $= (-\infty, 0) \cup (0, 2]$. $(g/f)(x) = \frac{4}{x\sqrt{2-x}}$, domain $= (-\infty, 0) \cup (0, 2)$ (note slightly different from the others). $(f \circ g)(x) = \sqrt{2 - \frac{4}{x}}$. (Was not asked for, but if you were curious, the domain is restricted to values of $x$ such that the "thing" inside the square root is not negative, so to find "bad" values we solve $2 - \frac{4}{x} < 0$. Solutions to that inequality are $(0, 2)$. Those are "bad" values. We also have bad values when we try to divide by zero, so $x = 0$ is not in the domain, either. So, the domain is everything else: $(-\infty, 0) \cup [2, \infty)$. $(g \circ f)(x) = \frac{4}{\sqrt{2-x}}$. (Domain is values such that $\sqrt{2-x} \neq 0$ and $2 - x \geq 0$, so $(-\infty, 2)$.)

4. Find $f^{-1}(x)$ if $f(x) = 1 + \frac{2}{3+x}$.

**Solution:** Set $y = f(x)$, swap $x$ and $y$, then solve for $y$ (alternatively, set $y = f(x)$, solve for $x$, then swap $x$ and $y$).

$$\begin{aligned}
y &= 1 + \frac{2}{3+x} \\
x &= 1 + \frac{2}{3+y} \\
x - 1 &= \frac{2}{3+y} \\
(x-1)(3+y) &= 2 \\
3 + y &= \frac{2}{x-1} \\
y &= \frac{2}{x-1} - 3 \\
f^{-1}(x) &= \frac{2}{x-1} - 3
\end{aligned}$$

5. Find $g^{-1}(x)$ if $g(x) = \frac{x-1}{2x+5}$.

**Solution:** As in the previous problem,

$$\begin{aligned}
y &= \frac{x-1}{2x+5} \\
x &= \frac{y-1}{2y+5} \\
x(2y+5) &= y-1 \\
2xy+5x &= y-1 \\
2xy-y &= -1-5x \\
y(2x-1) &= -1-5x \\
y &= \frac{-1-5x}{2x-1} \\
g^{-1}(x) &= \frac{-1-5x}{2x-1}
\end{aligned}$$

6. Let $f$ and $g$ be 1-1 functions satisfying

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 0 | 1 | 3 |
| 1 | 2 | 2 |
| 2 | 3 | 4 |
| 3 | 4 | 0 |

Evaluate the following:

(a) $(f \circ g)(0)$

(b) $(g \circ f)(0)$

(c) $f^{-1}(2)$

(d) $(g^{-1} \circ f)(1)$

(e) $(f^{-1} \circ g^{-1})(0)$

**Solution:**

(a) $(f \circ g)(0) = f(g(0)) = f(3) = 4$

(b) $(g \circ f)(0) = g(f(0)) = g(1) = 2$

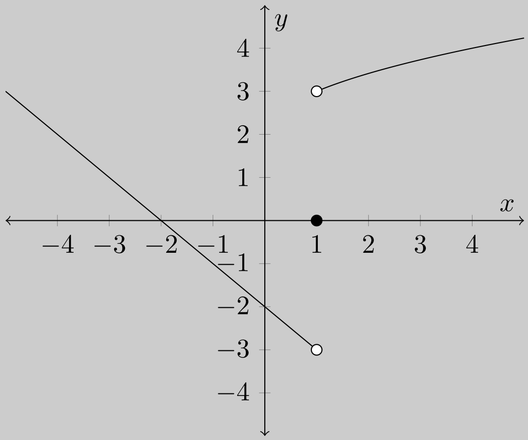
(c) $f^{-1}(2) = 1$, since $f(1) = 2$

(d) $g^{-1}(f(1)) = g^{-1}(2) = 1$, since $g(1) = 2$

(e) $(f^{-1} \circ g^{-1})(0) = f^{-1}(g^{-1}(0)) = f^{-1}(3) = 2$, since $g(3) = 0$ and $f(2) = 3$

7. Sketch a graph of the function $f(x) = \begin{cases} \sqrt{x}+2 & \text{if } x > 1 \\ 0 & \text{if } x = 1 \\ -x-2 & \text{if } x < 1 \end{cases}$, and state its domain and range.

**Solution:**




The domain of this function is $(-\infty, \infty)$, and the range is $(-3, \infty)$.

8. If $f(x) = (x-2)^2 + 1$, find a subset of the domain on which $f$ is 1-1, and on this subset, find a formula for $f^{-1}$.

**Solution:** Since $f(x)$ is a parabola, it is 1-1 if restricted to either all $x$ values to the left of the vertex, or all $x$ values to the right of the vertex. Here, the vertex is $(2, 1)$, so the possible subsets of the domain of $f$ are $(-\infty, 2]$ and $[2, \infty)$. For $f^{-1}$,

$$\begin{aligned} y &= (x-2)^2 + 1 \\ x &= (y-2)^2 + 1 \\ x - 1 &= (y-2)^2 \\ \pm\sqrt{x-1} &= y - 2 \\ y &= 2 \pm \sqrt{x-1} \\ f^{-1}(x) &= 2 \pm \sqrt{x-1} \end{aligned}$$

The positive square root is the partial inverse of $f$ along $[2, \infty)$, and the negative square root is the partial inverse of $f$ along $(-\infty, 2]$.

9. Repeat the previous problem for $g(x) = |x-3| - 4$.

**Solution:** Similar to the previous question, the graph of absolute value is shaped like a V, hence is 1-1 either to the left of the vertex or the right of the vertex. Here, the vertex is $(3, -4)$, so the possible restricted domains are $(-\infty, 3]$ and $[3, \infty)$. For $g^{-1}$, we can compute each case:

1. If $x$ is in $[3, \infty)$, then $|x-3| = x-3$, so $g(x) = (x-3) - 4 = x - 7$, and therefore, $g^{-1}(x) = x + 7$.
2. If $x$ is in $(-\infty, 3]$, then $|x-3| = -(x-3)$, so $g(x) = -(x-3) - 4 = -x - 1$, and therefore, $g^{-1}(x) = -(x+1) = -x - 1$.

10. Find the equation of the line through $(1, 2)$ and $(3, 4)$.

**Solution:** The slope is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{3-1} = \frac{2}{2} = 1$, so plugging in one of the points $(x, y)$ on the line, we get

$$\begin{aligned} y &= mx + b \\ y &= x + b \\ 2 &= 1 + b \\ b &= 1 \end{aligned}$$

It follows that the equation of this line is $y = x + 1$. Alternatively, find the slope and use point-slope form of the equation.

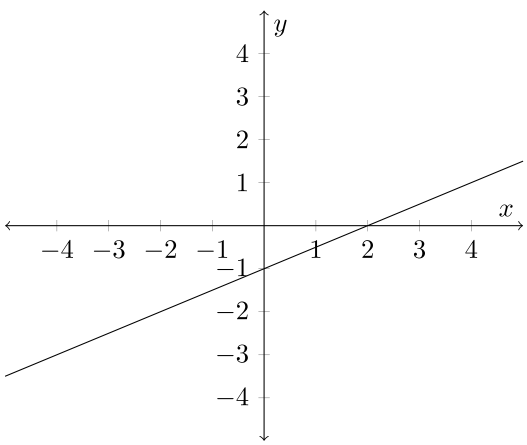
11. Find the equation of the line through $(5, 6)$ parallel to $y + 5 = 3x$.

**Solution:** Parallel lines have equal slopes, so since $y + 5 = 3x$ has slope 3, our line has $m = 3$ as well. As in the previous problem,

$$\begin{aligned} y &= mx + b \\ y &= 3x + b \\ 6 &= (3)(5) + b \\ b &= 6 - 15 = -9 \end{aligned}$$

The equation of our line is $y = 3x - 9$.

12. Find the equation of the line below.




**Solution:** The $y$-intercept is -1, and the slope is $\frac{1}{2}$, since the line moves exactly up 1 and over 2 from $(0, -1)$ to $(2, 0)$. That makes the equation of this line $y = \frac{1}{2}x - 1$. Alternatively, use the points $(0, -1)$ and $(2, 0)$ and proceed as in question 16.

13. Find the equation of the line through $(-3, 2)$ perpendicular to $y = 5x + 7$.

**Solution:** Since the slopes of perpendicular lines are negative reciprocals of each other, $y = 5x + 7$ having slope 5 means our line has slope $-\frac{1}{5}$.

$$\begin{aligned} y &= mx + b \\ y &= -\frac{1}{5}x + b \\ 2 &= -\frac{1}{5}(-3) + b \\ b &= 2 - \frac{3}{5} = \frac{7}{5} \end{aligned}$$

The equation of our line is $y = -\frac{1}{5}x + \frac{7}{5}$.

14. For each of the quadratic functions below, do five things:
    1. find its roots, if any
    2. find its y-axis intercept
    3. complete the square
    4. graph it
    5. determine its range

    (a) $x^2 - 4x$

    (b) $x^2 + 10x - 4$

    (c) $-x^2 + 6x + 2$

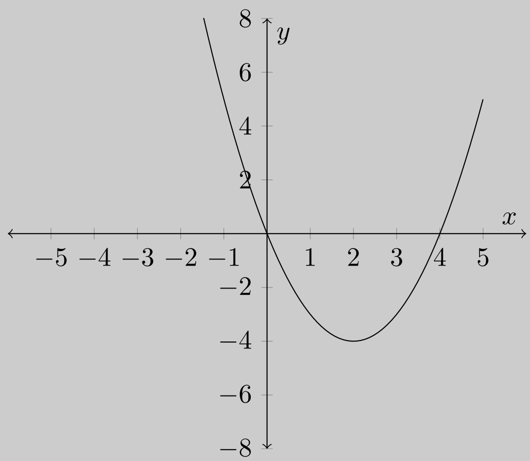
    (d) $x^2 + 5x + 1$

    (e) $3x^2 - 9x + 12$

    (f) $2x^2 + 3x + 4$

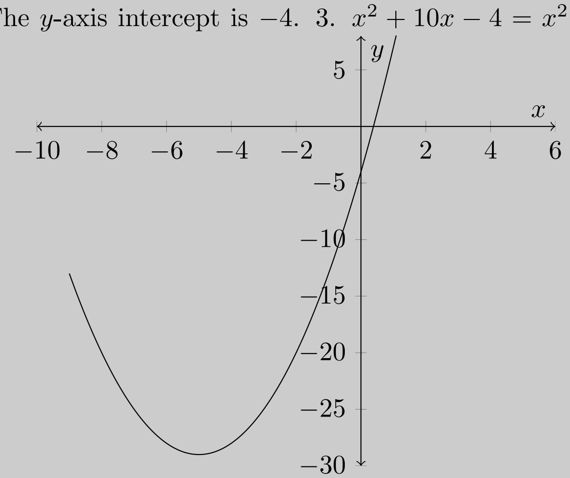
**Solution:**

(a) 1. $x^2 - 4x = 0 \implies x(x-4) = 0 \implies x = 0, 4$, so the roots are 0 and 4. 2. The $y$-axis intercept is $f(0) = 0$. 3. $x^2 - 4x = x^2 - 4x + 4 - 4 = (x-2)^2 - 4$. 4.




5. The range is $[-4, \infty)$.

(b) 1. $x^2 + 10x - 4 = 0$...perhaps QF is best. $x = \frac{-10 \pm \sqrt{100 - 4(1)(-4)}}{2}$ are the roots. 2. The $y$-axis intercept is $-4$. 3. $x^2 + 10x - 4 = x^2 + 10x + 25 - 25 - 4 = (x+5)^2 - 29$.




4.

5. The range is $[-29, \infty)$.

(c) (Abbreviated solution) Complete the square: $-x^2+6x+2=-(x^2-6x-2)=-(x^2-6x+9-9-2)=-((x-3)^2-11)=-(x-3)^2-11$, so the vertex of this parabola is at $(3,-11)$, and it faces downward. Thus, it has no roots, and its range is $(-\infty,-11]$. Graph omitted.

(d) (Abbreviated solution) Roots: $\frac{-5\pm\sqrt{25-4(1)(1)}}{2}$. Complete the square: $x^2+5x+1=x^2+5x+\frac{25}{4}-\frac{25}{4}+1=(x+\frac{5}{2})^2-\frac{21}{4}$. Range: $[-\frac{21}{4},\infty)$.

(e) (Abbreviated solution) Complete the square: $3x^2-9x+12=3(x^2-3x+4)=3(x^2-3x+\frac{9}{4}-\frac{9}{4}+4)=3((x-\frac{3}{2})^2+\frac{7}{4})=3(x-\frac{3}{2})^2+\frac{21}{4}$. No roots.

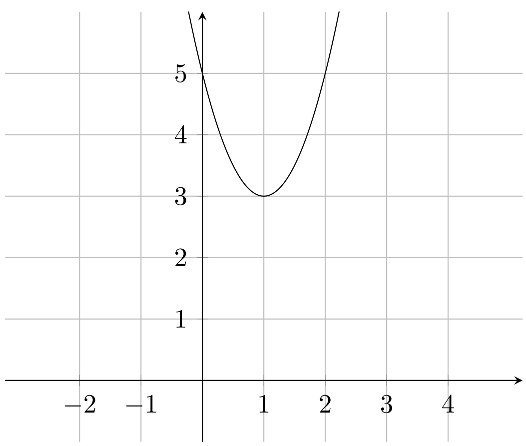
(f) (Abbreviated solution) Complete the square: $2x^2+3x+4=2(x^2+\frac{3}{2}x+2)=2(x^2+\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}+2)=2((x+\frac{3}{4})^2+\frac{23}{16})=2(x+\frac{3}{4})^2+\frac{23}{8}$. No roots.

15. Find a function $f$ whose graph is a parabola with a vertex at $(1,3)$ such that $f(2)=7$.

**Solution:**

If the vertex is at $(1,3)$, that means it is a parabola shifted to the right by 1 and up by 3, so it's something like $f(x)=a(x-1)^2+3$. We just need to find $a$. However, we also know that $f(2)=7$, which means that $a(2-1)^2+3=7$, so $a=4$. Therefore, $f(x)=4(x-1)^2+3$.

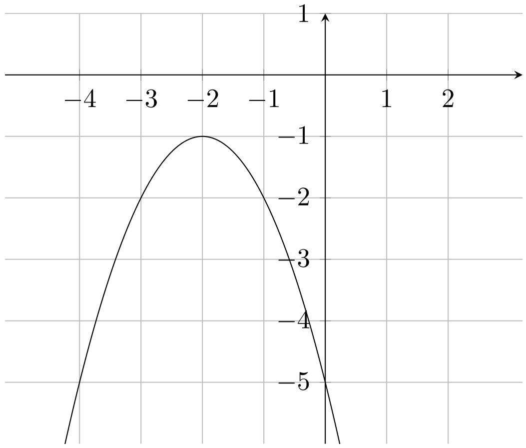
16. What is the equation of the parabola below?




**Solution:**

The vertex is at $(1,3)$, which means it must be something of the form $p(x)=a(x-1)^2+3$. We just don't know what $a$ is but we can use the information that the parabola passes through $(0,5)$; that is, if we plug in $x=0$ we should get $p(x)=5$. That is, $a(0-1)^2+3=5$, or in other words, $a=2$. So $p(x)=2(x-1)^2+3$.

17. What is the equation of the parabola below?

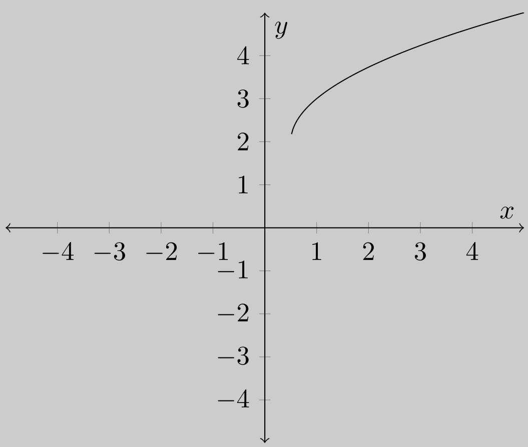



**Solution:**

The vertex is at $(-2,-1)$ so the form is $f(x) = a(x+2)^2 - 1$. The parabola passes through $(0,-5)$ so $f(0) = -5$, or in other words, $-5 = a(0+2)^2 - 1$, so $a = -1$. Therefore, $f(x) = -(x+2)^2 - 1$.

18. Sketch a graph of the function $g(x) = \sqrt{2x-1} + 2$, and label three points on the graph, in addition to its axis intercepts.

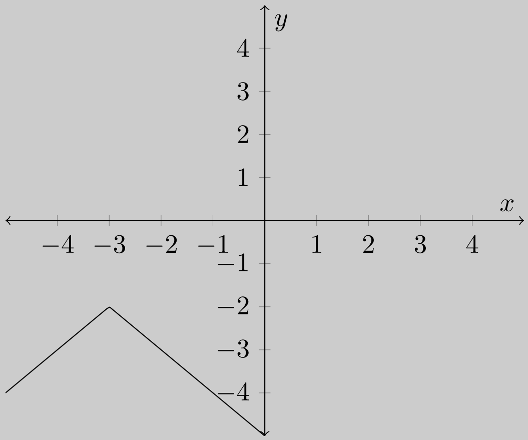
**Solution:**




The graph is that of $\sqrt{x}$, except shifted right by 1, then horizontally compressed by a factor of 2, and then shifted up by 2. Three points on the graph could be $(\frac{1}{2}, 2)$, $(1,3)$, and $(3, \sqrt{5}+2)$. It has no axis intercepts.

19. Sketch a graph of the function $f(x) = -|x+3| - 2$, and state its range.

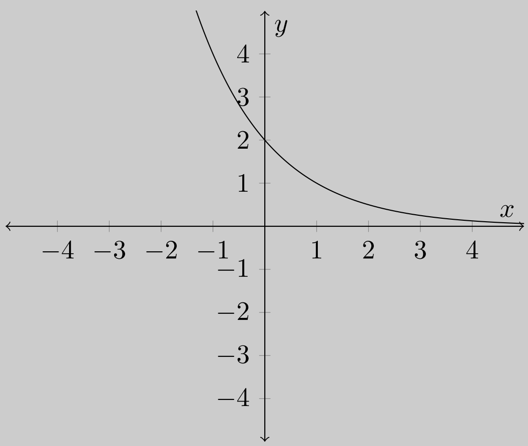
**Solution:**




The graph is shifted left by 3, then flipped over the $x$-axis, then shifted down by 2. The range is $(-\infty, -2]$.

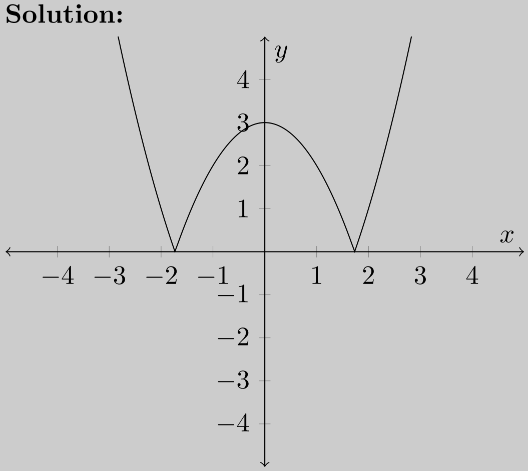
20. Sketch a graph of the function $f(x) = 2^{1-x}$.

**Solution:**

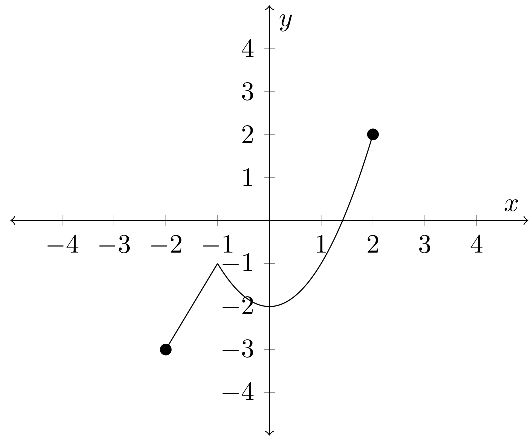



21. Sketch a graph of the function $h(x) = |x^2 - 3|$.

**Solution:**

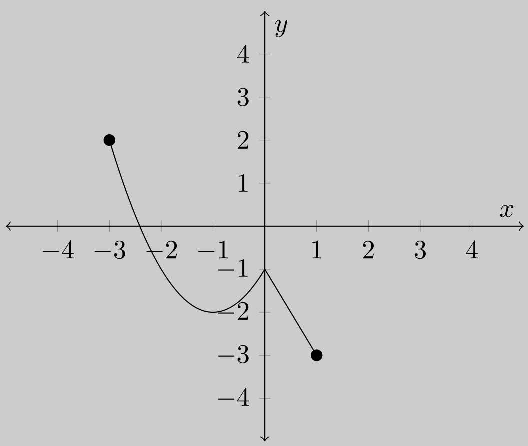



22. Given the graph of $f$ below, sketch the graphs of $f(-x-1)$, $-f(x)-1$, and $2f(2x)$.

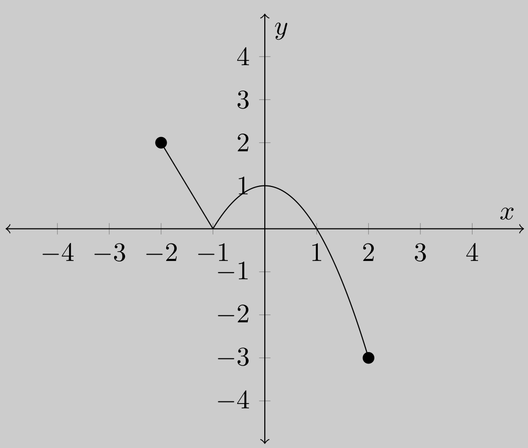



**Solution:**

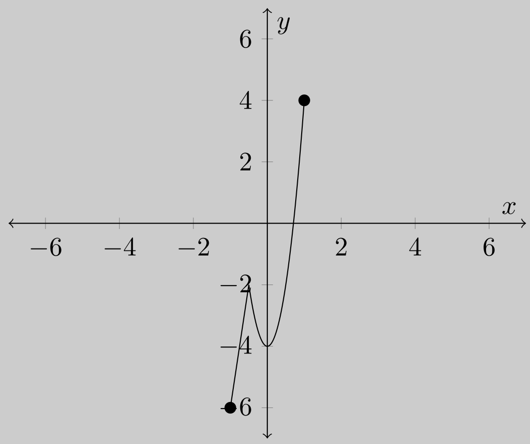
$f(-x-1)$ is reflected over the $y$-axis and then shifted left by 1 (or alternatively, shifted right by 1 and then reflected).




$-f(x)-1$ is reflected over the $x$-axis and then shifted down by 1.




$2f(2x)$ is compressed by a factor of 2 in the horizontal direction, and stretched by a factor of 2 in the vertical direction.




23. Calculate $(3x^3 - 2x^2 + 4x - 3) \div (x^2 + 3x + 3)$.

**Solution:**

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad 3x - 11 \\ x^2+3x+3 & 3x^3 - 2x^2 + 4x - 3 \\ & -3x^3 - 9x^2 - 9x \\ \hline & \quad -11x^2 - 5x - 3 \\ & \quad\ \ 11x^2 + 33x + 33 \\ \hline & \quad\quad\quad\quad 28x + 30 \end{array}$$

24. Calculate $(20x^3 + 16x^2 + 26x - 9) \div (5x - 1)$.

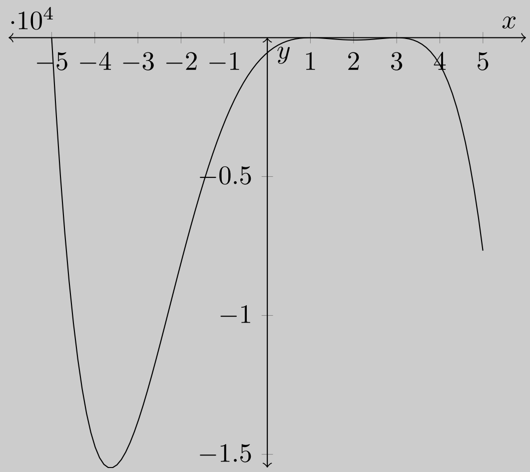
**Solution:**

$$\begin{array}{r|l} & \quad\quad 4x^2 + 4x + 6 \\ 5x-1 & 20x^3 + 16x^2 + 26x - 9 \\ & -20x^3 + 4x^2 \\ \hline & \quad\quad 20x^2 + 26x \\ & \quad -20x^2 + 4x \\ \hline & \quad\quad\quad\quad 30x - 9 \\ & \quad\quad\quad -30x + 6 \\ \hline & \quad\quad\quad\quad\quad -3 \end{array}$$

25. What is the degree and the leading coefficient of the polynomial $-3(x-3)^2(x+5)(2x-2)^4$? Determine its end behavior. Sketch a graph of the polynomial.

**Solution:**

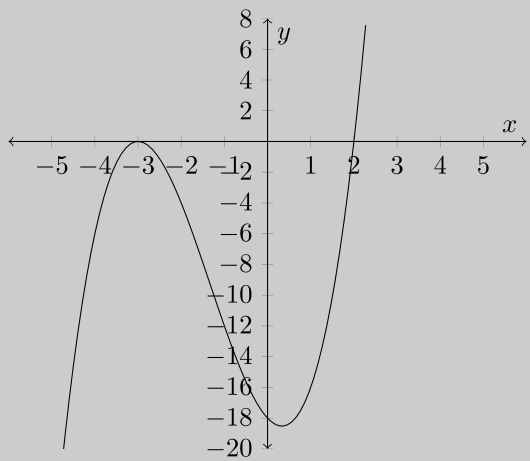
The leading term of the polynomial must come from multiplying the leading term of each factor of the polynomial. That is, we multiply the terms: $\mathbf{-3(x}-3)^{\mathbf{2}}(\mathbf{x}+5)(\mathbf{2x}-2)^{\mathbf{4}}$ to get $-3 \cdot x^2 \cdot x \cdot (2x)^4 = -3 \cdot x^2 \cdot x \cdot 16x^4 = -48x^7$. So the degree is 7, the leading coefficient is -48, and as $x \to \infty, f(x) \to -\infty$, and as $x \to -\infty, f(x) \to \infty$. The graph looks something like this (the numbers are very large so the scale is off...but we're just looking for the general shape):




26. Sketch a graph of the polynomial $f(x) = (x-2)(x^2+6x+9)$.

**Solution:**

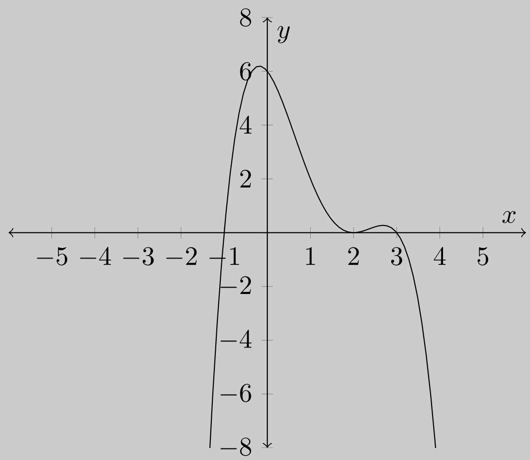
(Only a rough graph would be expected, showing zeros, positive/negative regions, and end behavior.)




27. Sketch a graph of the polynomial $g(x) = -\frac{1}{2}(x-3)(x+1)(x-2)^2$.

**Solution:**

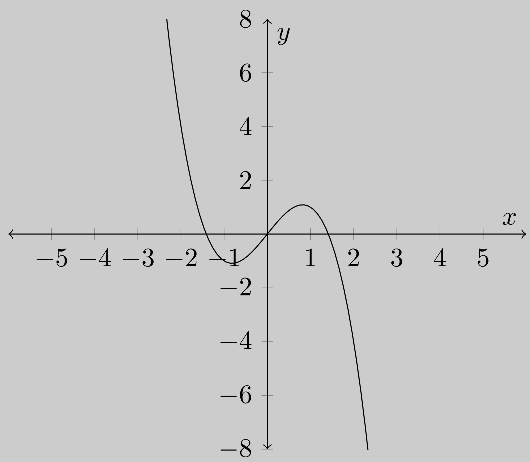
(Only a rough graph would be expected, showing zeros, positive/negative regions, and end behavior.)




28. Sketch a graph of the polynomial $h(x) = -x^3 + 2x$.

**Solution:**

(Only a rough graph would be expected, showing zeros, positive/negative regions, and end behavior.)




29. Use the Intermediate Value Theorem to show that the polynomial $p(x) = x^{94} - 3x^2 + 1$ has a root in the interval $[-2, 1]$.

**Solution:** Since $p(-2)$ =something really big and positive and $p(1) = -1$ and all polynomials are continuous, the IVT states that there must be a value $c$ in the interval $(-2, 1)$ such that $p(c) = 0$.

30. True or false? If $f(x) = \frac{3}{x-2}$, since $f(-2) = -\frac{3}{4}$ and $f(3) = 3$, by the Intermediate Value Theorem, $f$ must have a root in $(-2, 3)$.

**Solution:** False – we can't use the IVT because $f$ is not continuous.

31. Compute or simplify the following:
    (a) $\log_3 3$
    (b) $\log_7 1$
    (c) $\log_{25} 5$

(d) $\log_{25}\frac{1}{5}$

(e) $\log_{25} -5$

(f) $\log_{25} 125$

(g) $\log_{25} 0$

(h) $\log_4 2^{37}$

(i) $\log_2 4^{37}$

(j) $3^{\log_3 9}$

(k) $3^{\log_2 16}$

(l) $e^{\ln x}$

(m) $e^{\ln x^2+5}$

(n) $\ln e^2$

(o) $\ln e^0$

**Solution:** It often helps to rewrite these in exponential form. Remember, $\log_a(x) = y$ is the same as saying $a^y = x$.

(a) 1

(b) 0

(c) $\frac{1}{2}$

(d) $-\frac{1}{2}$

(e) DNE

(f) $\frac{3}{2}$

(g) DNE

(h) $\log_4 2^{37} = x$ if $4^x = 2^{37}$. Since $2^{37} = 4^{\frac{37}{2}}$, $x = \frac{37}{2}$.

(i) $\log_2 4^{37} = x$ if $2^x = 4^{37}$. Since $4^{37} = (2^2)^{37} = 2^{74}$, $x = 74$.

(j) 9

(k) $3^4 = 81$.

(l) $x$

(m) $x^2 + 5$

(n) 2

(o) 0

32. Does the function $x^3 - x$ have an inverse?

**Solution:** Nope. Notice that $x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$. In other words, this function has three roots, $x = 0, -1, 1$, i.e., $f(0) = f(-1) = f(1)$. Therefore, it's not 1-1, so it can't have an inverse.

33. Find an inverse for the function $f(x) = 2\log_3 x$.

**Solution:** Let $x = 2\log_3 y$. Then:

$$\begin{aligned} x &= 2\log_3 y \\ \frac{x}{2} &= \log_3 y \\ 3^{\frac{x}{2}} &= y \end{aligned}$$

So $f^{-1}(x) = 3^{\frac{x}{2}}$.

34. Find an inverse for the function $g(x) = \frac{e^{x+4}}{2}$.

**Solution:** Let $x = \frac{e^{y+4}}{2}$. Then:

$$\begin{aligned} x &= \frac{e^{y+4}}{2} \\ 2x &= e^{y+4} \\ \ln(2x) &= y + 4 \\ \ln(2x) - 4 &= y \end{aligned}$$

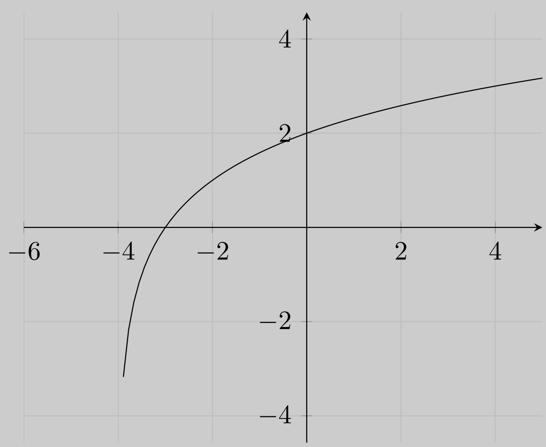
So $g^{-1}(x) = \ln(2x) - 4$.

35. What is the domain of the function $\ln|x|$? Can you find an inverse for this function?

**Solution:** We can plug in any positive number to the natural logarithm function. Since $|x|$ is positive unless $x = 0$, we can plug in any number except for 0, i.e., the domain is $(-\infty, 0) \cup (0, \infty)$. This function is not $1-1$, however, since $\ln|-3| = \ln|3|$, for example. Therefore, we can't find an inverse. (A partial inverse would be $e^x$.)

36. Sketch a graph of $\log_2(x + 4)$.

**Solution:** You should know the general shape of the graph of a logarithm function, including where it crosses the $x$-axis. This one is shifted 4 units to the left.




37. Solve for $x$: $3\ln(x) - 2 = 4$.

**Solution:**

$$\begin{aligned} 3\ln(x) - 2 &= 4 \\ 3\ln(x) &= 6 \\ \ln(x) &= 2 \\ x &= e^2 \end{aligned}$$

38. Solve for $x$: $\log_2 \frac{1}{2} + 3 = \frac{1}{x}$.

**Solution:**

$$\begin{aligned} \log_2 \frac{1}{2} + 3 &= \frac{1}{x} \\ -1 + 3 &= \frac{1}{x} \\ 2 &= \frac{1}{x} \\ x &= \frac{1}{2} \end{aligned}$$

39. Simplify $\ln(e^{\ln(e^x)})$.

**Solution:**

$$\ln(e^{\ln(e^x)}) = \ln(e^x) = x$$

40. Given the function $f(x) = \dfrac{x^2 - 9}{x(x-3)}$:

(a) Find any vertical asymptotes of $f$.

(b) Find any horizontal asymptotes of $f$.

(c) Find any zeros of $f$.

(d) What is the domain of $f$?

(e) Sketch a graph of $f$.

**Solution:** Note that we can simplify $f$ as follows:

$$f(x) = \frac{(x-3)(x+3)}{x(x-3)} = \frac{x+3}{x}.$$

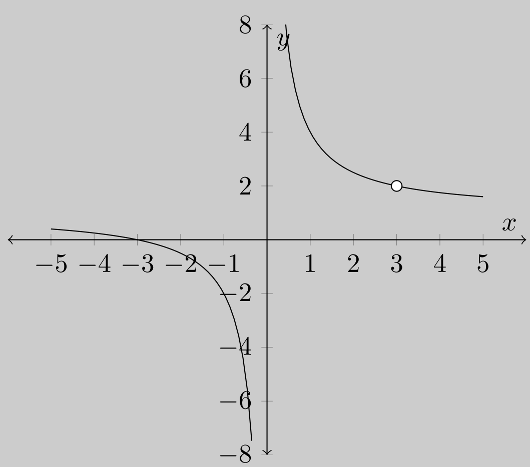
(a) $f$ has a vertical asymptote at $x = 0$

(b) $f$ has a horizontal asymptote at $y = 1$

(c) $f$ has a zero at $x = -3$

(d) The domain of $f$ is $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$ (i.e., $f$ has a hole at 3)

(e) $f$ looks like:



41. Given the function $f(x) = \dfrac{(x+1)(x-4)}{(x+4)(x+1)(x+7)}$:

   (a) Find any vertical asymptotes of $f$.

   (b) Find any horizontal asymptotes of $f$.

   (c) Find any zeros of $f$.

   (d) What is the domain of $f$?

   (e) Sketch a graph of $f$.

**Solution:** Note that we can simplify $f$ as follows:

$$f(x) = \frac{x-4}{(x+4)(x+7)}.$$

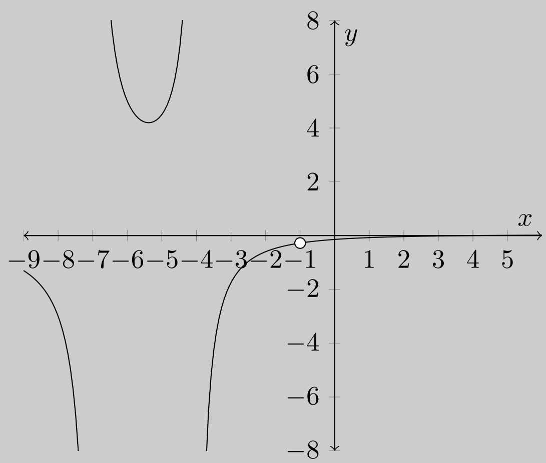
(a) $f$ has a vertical asymptote at $x = -4$ and $x = -7$.

(b) $f$ has a horizontal asymptote at $y = 0$

(c) $f$ has a zero at $x = 4$

(d) The domain of $f$ is $(-\infty, -7) \cup (-7, -4) \cup (-4, -1) \cup (-1, \infty)$

(e) $f$ looks like:



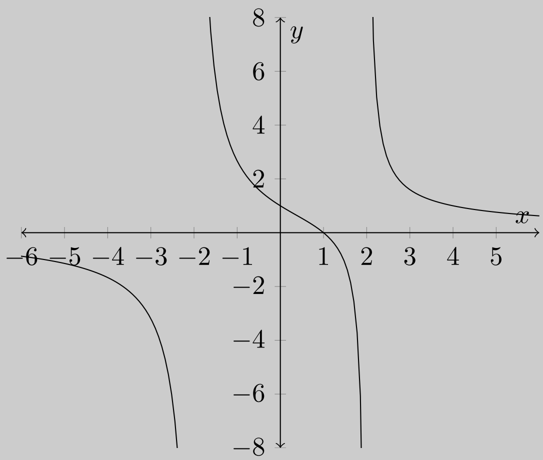

42. Graph the function $f(x) = \frac{1}{x-2} + \frac{3}{x+2}$, clearly indicating any zeros, holes, and asymptotes.

**Solution:**

In this case, we can rewrite $f$ as:

$$f(x) = \frac{1}{x-2} + \frac{3}{x+2} = \frac{(x+2)+(3x-6)}{(x-2)(x+2)} = \frac{4(x-1)}{(x-2)(x+2)}.$$

Therefore, $f$ has no holes, has a vertical asymptote at $x = 2$ and $x = -2$, has a horizontal asymptote at $y = 0$, and has a zero at $x = 1$. A little more work plugging in numbers figures out which way the function goes at each asymptote (to positive or negative infinity).




43. Does the function $f(x) = \frac{3x-6}{2^x}$ have any vertical asymptotes?

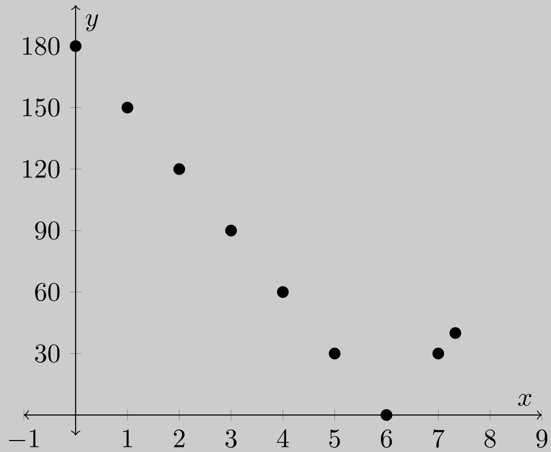
**Solution:** The function $f$ is defined everywhere, since $2^x$ is never equal to 0, so it has no vertical asymptotes.

44. Does the function $f(x) = \frac{3}{2^x-8}$ have any vertical asymptotes? Horizontal asymptotes?

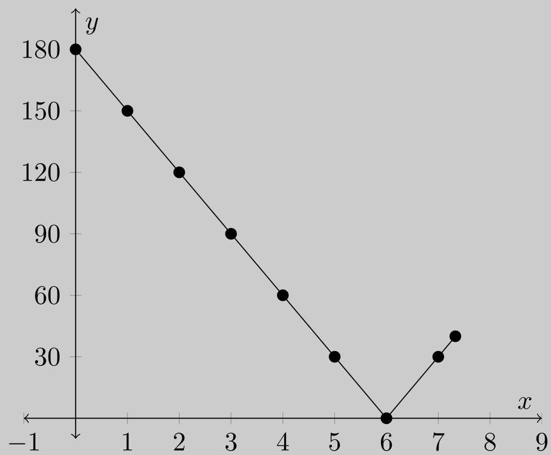
**Solution:** When $x = 3$, the denominator will equal 0, and the numerator will be 3. In other words, for values of $x$ near 3, the numerator will be 3, and the denominator will get infinitesimally small, meaning the function value will get arbitrarily large – a vertical asymptote. As $x$ approaches infinity, the denominator will get very, very big, while the numerator will always be 3, meaning the function value approaches 0 – a horizontal asymptote. As $x$ approaches negative infinity, the denominator will approach $-8$, and the numerator will be 3, so the function value will get closer and closer to $-\frac{3}{8}$, which is another horizontal asymptote.

45. Kyle is driving from City A to City C at a constant rate of 30 mph. In between, is City B. City A is 180 miles from City B, and City B is 40 miles from City C. Write Kyle's distance from City B as a function of time.

**Solution:** You could plot some points. At time 0, Kyle is 180 miles from City B. At time 1 hour, Kyle is 150 miles from City B. At 2 hours, he's 120 miles. And so on, so points on the graph of our function are $(0, 180), (1, 150), (2, 120), (3, 90), (4, 60), (5, 30), (6, 0)$. Then Kyle passes through City B and starts getting farther away, so at 7 hours, he's 30 miles away again, i.e., $(7, 30)$ is also on our graph. When does he reach City C? He needs another 10 miles, which should take him 1/3 of an hour, so at 7.33 hours, he's 40 miles away, i.e., $(7.33, 40)$ is on the graph. Go ahead and plot these points – what does it look like?



Looks an awful lot like a shifted and scaled absolute value function. So it could be something like $f(x) = a|bx - c|$, where we have to figure out the values of $a, b$, and $c$. The $y$ values change at a rate of 30 units for every $x$ unit, so let's try $a = 30$. And, we're shifted 6 units to the right, so let's try $c = 6$. Maybe we don't need $b$? Let's graph $f(x) = 30|x - 6|$:



Perfect!

46. The length of a rectangle is six feet more than twice its width. Its area is 28. Find the dimensions of the rectangle.

**Solution:**

Let's say the length of the rectangle is $x$ and the width is $y$. Then the length, $x$, is 6 feet *more* than two times $y$; that is, if we take $2y$ and add 6 we should get $x$, i.e., $2y + 6 = x$. The area is 28, or in other words, $xy = 28$. That means that $x = \frac{28}{y}$, so $2y + 6 = \frac{28}{y}$. Now we just have an equation to solve. Multiply both sides by $y$ to get $2y^2 + 6y - 28 = 0$, and solve by factoring or using the quadratic formula. After some trying we find this one is hard to factor, so we use the QF: $\frac{-6 \pm \sqrt{36+4\cdot 2\cdot(-28)}}{4}$. To have a reasonable (positive) answer, we must use the positive root, so the length of one side is $\frac{-6+\sqrt{36+4\cdot 2\cdot(-28)}}{4}$.

47. The concentration of a certain antibiotic in a patient's bloodstream is given by $C(t) = \frac{25t}{t^2+600}$, where $t$ represents the number of hours since the antibiotic was taken.

    1. Explain the meaning of the horizontal asymptote of this function.
    2. What is the concentration after 3 hours on the antibiotic?
    3. The antibiotic is only effective if its concentration is above 0.5. When must the patient take another dose?

**Solution:**

1. The horizontal asymptote tells us what happens as $t \to \infty$, so it represents the level of antibiotic in the bloodstream after lots of time has passed.
2. $C(3) = \frac{75}{609}$.
3. For what value of $t$ is $C(t) > 0.5$? We can solve $0.5 < \frac{25t}{t^2+600}$:

$$\begin{aligned} \frac{t^2}{2} + 300 &< 25t \\ \frac{t^2}{2} - 25t + 300 &< 0 \\ t^2 - 50t + 600 &< 0 \\ (t-20)(t-30) &< 0 \end{aligned}$$

This inequality is true if $t$ is between 20 and 30. So at time 0, the patient has no antibiotic, it is injected and rises to above 0.5 at time $t = 20$, and then by time $t = 30$ it has decreased again lower than 0.5 and the patient must take another dose.

48. You could see a few conceptual questions on the exams as well, i.e., questions that require written explanation or critical thought, rather than calculation. Here are a few examples.

(a) Explain the difference between $f(x) = \frac{x^2-2x+1}{x-1} = \frac{(x-1)(x-1)}{x-1}$ and $g(x) = x - 1$.

(b) Explain why a function needs to be one-to-one in order to have an inverse. If you like, you can use the function $f(x) = |x|$ as an example.

(c) Explain why the vertical line test tells you if a graph represents the graph of a function or not.

(d) If the range of $f$ is $[0, \infty)$ and the domain of $f$ is $(-\infty, \infty)$, what is the domain of $f^{-1}$?

(e) Is $f(x) = \sqrt{x}$ the inverse of $g(x) = x^2$?

(f) Complete the following definition: a function $f$ is increasing if, for all $x_1 < x_2, \ldots$

(g) Let $f(x) = \sqrt{x}$. Then as $x \to \infty$, $\sqrt{x} \to ???$.

(h) Define the function $\log_2 x$.

(i) True or false? If a function passes the horizontal line test, then so does its inverse.

(j) Give an example of a function whose domain is $(1, \infty)$ and whose range is $(-\infty, \infty)$.

(k) Which is a bigger number, $\ln(10)$ or $\log_{10}(e)$?

(l) In order for 3 to be in the domain of $f \circ g$, must it be true that 3 is in the domain of $f$?

**Solution:** Not all of these are full answers, but they're intended to get you started thinking in the right way.

(a) The domains of the functions are different (how?).

(b) It has to do with the fact that a function can only have one output for each input.

(c) It has to do with the fact that a function can only have one output for each input.

(d) The domain of $f$ is the range of $f^{-1}$, and vice versa.

(e) Not exactly. Remember, $g(x) = x^2$ is 1-1.

(f) $\ldots f(x_1) < f(x_2)$.

(g) $\infty$.

(h) The function $\log_2 x$ is the inverse of the function $2^x$.

(i) True (why?).

(j) $\ln(x-1)$

(k) $\ln(10)$ (why?)

(l) Nope (why?)

49. Want more questions? Check out the Learning Objectives in the Course Outline. Each week's set of learning objectives contains some sample problems as well. These sample problems won't have solutions, but you can talk to your instructor, fellow students, or the Q Center to get solutions.